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## Non-integrability of the mixmaster universe

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**Abstract.** We comment on an analysis by Contopoulos *et al* which demonstrates that the governing six-dimensional Einstein equations for the mixmaster spacetime metric pass the ARS or reduced Painlevé test. We note that this is the case irrespective of the value,  $I$ , of the generating Hamiltonian which is a constant of motion. For  $I < 0$  we find numerous closed orbits with two unstable eigenvalues strongly indicating that two additional first integrals apart from the Hamiltonian cannot exist and thus that the system, at least for this case, is very probably not integrable. In addition, we present numerical evidence that the average Lyapunov exponent nevertheless vanishes.

The model is thus a very interesting example of a Hamiltonian dynamical system, which is probably non-integrable yet passes the reduced Painlevé test.

### 1. Introduction

In general, it is a difficult task to establish whether or not a Hamiltonian system of  $N$  degrees of freedom is integrable. We note that there are many definitions of integrability [1]. Here, we take as definition [2] the existence of  $N$  independent first integrals  $I_j = I_j(\mathbf{p}, \mathbf{q}) = \text{constant}$  which are *in involution*, i.e. their mutual Poisson brackets vanish  $\{I_i, I_j\} = 0$ , for all  $i, j$ . If  $N$  such integrals have not been found, one may try to establish the integrability of the dynamical system via the singularity analysis called the Painlevé test. According to the ‘Painlevé conjecture’ a Hamiltonian dynamical system is integrable if it has the Painlevé property. However, since carrying out the Painlevé test is often too difficult a task, in practice the test is performed in a reduced form called the ARS algorithm [3–5]. In this paper we shall comment on an example where the ARS conjecture fails.

In Contopoulos *et al* [6], and in Cotsakis and Leach [7], it is shown that the Einstein equations for the mixmaster spacetime metric pass the ARS test (the reduced Painlevé property) and, by the above proposition, that it is probably integrable.

Here we shall argue that this is very probably *not the case*. Our argument pursues the following line. First, there is a generating Hamiltonian, the value of which,  $I$ , characterizes different qualitative behaviours of the model. The case  $I = 0$  corresponds to the proper mixmaster universe. Although it contains the chaotic Gauss map in its time evolution there is no direct contradiction in calling the system integrable, e.g. in the sense of being solvable by an inverse scattering transform (see e.g. [3]). For  $I > 0$ , the evolution is in some sense

'trivial', but for  $I < 0$ † we shall show that the system displays chaotic behaviour in the sense that it contains an abundance of unstable periodic orbits, even apparently infinite families of these, which in turn renders the existence of a full set of first integrals very unlikely. In fact, the proliferation of unstable periodic orbits indicates the existence of a 'Smale horseshoe' prohibiting integrability. A rigorous proof would require finding a transversal homoclinic intersection [10, 11], i.e. a transversal intersection of the stable and unstable manifolds of a hyperbolic closed orbit of the system, a work currently in progress. Second, the Painlevé analysis for the mixmaster model as presented in Contopoulos *et al* [6] *does not* distinguish the value of this first integral, i.e. by this analysis one arrives at the contradictory statement that even in the chaotic case  $I < 0$  the model is probably integrable‡.

Finally, we present numerical evidence that, despite the existence of unstable periodic orbits, the average Lyapunov exponent vanishes in the case  $I < 0$ , contradicting data based on single trajectories presented in Rugh [8] and Hobill *et al* [14].

## 2. The equations of motion

The mixmaster spacetime metric is a famous cosmological toy model [15] which has been studied extensively in the past two decades in various contexts. It is an anisotropic generalization of the standard cosmological model of our universe (in case our universe is closed), the compact Friedman–Robertson–Walker model. The mixmaster metric exhibits very complicated dynamical behaviour near its curvature singularities, illustrating an interesting nonlinear aspect of the Einstein equations. For a recent discussion of the characterization of chaos in general relativity and the mixmaster toy-model gravitational collapse, see e.g. Rugh [16].

The governing Einstein equations for the mixmaster metric are given by a set of three second-order ordinary differential equations

$$\begin{aligned} 2\ddot{\alpha} &= (e^{2\beta} - e^{2\gamma})^2 - e^{4\alpha} \\ 2\ddot{\beta} &= (e^{2\gamma} - e^{2\alpha})^2 - e^{4\beta} \\ 2\ddot{\gamma} &= (e^{2\alpha} - e^{2\beta})^2 - e^{4\gamma} \end{aligned} \quad (1)$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are the so-called scale factors of the metric, and a dot denotes derivative with respect to the (logarithmic) time variable  $\tau$ , see Landau and Lifshitz [15] section 118. Equations (1) admit a first integral

$$I = 4(\dot{\alpha}\dot{\beta} + \dot{\beta}\dot{\gamma} + \dot{\gamma}\dot{\alpha}) - e^{4\alpha} - e^{4\beta} - e^{4\gamma} + 2(e^{2(\alpha+\beta)} + e^{2(\beta+\gamma)} + e^{2(\gamma+\alpha)}). \quad (2)$$

The *mixmaster universe*, with its interesting oscillatory behaviour of the scale functions near its spacetime singularity, arises exactly at  $I = 0$ . The cosmological model was first analysed by Belinski and Khalatnikov [17] and independently by Misner [18], who showed that the asymptotic evolution could be accurately described as a simple combinatorial evolution of the axes governed by the Gauss map (see also Barrow [19] and Mayer [20]). The appearance of the Gauss map may seem to indicate chaotic behaviour of the system but, as the transformation to the relevant coordinates is singular and maps non-closed orbits

† At first sight a universe model with  $I < 0$  might appear quite artificial. It has, however, a physical interpretation as a mixmaster universe in which matter with negative energy density is uniformly distributed. It gives the possibility of an oscillatory evolution of the three-volume of the universe [8, 9] and thus the possibility of periodic (*cyclic*) *mixmaster universes*.

‡ In a subsequent paper [12] Contopoulos *et al* cast doubt as to the conclusion of their original paper [6]. They introduce a *perturbative* Painlevé test which the mixmaster model fails to pass. Also, Latifi *et al* [13] note that the model does not pass the full Painlevé test.

into closed orbits, it is not clear to what extent conclusions regarding the inherent chaos of the Gauss map can be carried back to the analysis of the differential system governing the mixmaster universe.

Following the notation of Contopoulos *et al* [6], we first perform a change of variables:

$$\begin{aligned} x &= 2\alpha & y &= 2\beta & z &= 2\gamma \\ p_x &= -(\dot{y} + \dot{z}) & p_y &= -(\dot{z} + \dot{x}) & p_z &= -(\dot{x} + \dot{y}) \end{aligned} \tag{3}$$

and find that the system can also be described as governed by the Hamiltonian

$$\begin{aligned} H &= \frac{1}{4}(p_x^2 + p_y^2 + p_z^2 - 2p_x p_y - 2p_y p_z - 2p_z p_x) \\ &\quad + e^{2x} + e^{2y} + e^{2z} - 2e^{x+y} - 2e^{y+z} - 2e^{z+x} \equiv -I. \end{aligned} \tag{4}$$

For the purpose of the Painlevé analysis a further change of variables is made:

$$X = e^x \quad Y = e^y \quad Z = e^z \tag{5}$$

and we finally end up with this version of differential equations:

$$\begin{aligned} 2\dot{X} &= X(p_x - p_y - p_z) & 2\dot{Y} &= Y(p_y - p_z - p_x) & 2\dot{Z} &= Z(p_z - p_x - p_y) \\ \dot{p}_x &= 2X(Y + Z - X) & \dot{p}_y &= 2Y(Z + X - Y) & \dot{p}_z &= 2Z(X + Y - Z) \end{aligned} \tag{6}$$

in which the first integral takes the form

$$I = -\frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{4}(p_x + p_y + p_z)^2 - 2(X^2 + Y^2 + Z^2) + (X + Y + Z)^2. \tag{7}$$

The sign of the first integral characterizes three different phases of the model. This can be seen from the scale invariance of the differential equations (6) under the transformation

$$\begin{aligned} X &\rightarrow cX & Y &\rightarrow cY & Z &\rightarrow cZ \\ p_x &\rightarrow cp_x & p_y &\rightarrow cp_y & p_z &\rightarrow cp_z \\ I &\rightarrow c^2 I & \tau &\rightarrow c^{-1}\tau. \end{aligned} \tag{8}$$

In popular terms we may say that  $I = 0$  characterizes a phase transition for the system, as the qualitative description of the time evolution depends strongly on the sign of  $I$  [8]. In particular we note here that since the time variable scales with  $|I|^{-1/2}$ , the Lyapunov exponent for the system will have to vanish for  $I \rightarrow 0$ , revealing an apparent non-chaotic behaviour at  $I = 0$ , i.e. for the mixmaster universe. Due to this scale invariance we are free to choose an arbitrary value of  $I$  when we look for periodic orbits of the mixmaster system for  $I < 0$ , and we will choose the value  $I = -1$ .

### 3. Painlevé analysis

The Painlevé analysis, when applied to (6), showed [6] that the system passes the ARS Painlevé test for the case  $I = 0$  and, therefore, according to the cited conjecture, that the system is probably integrable. The analysis does not, however, depend on the value of  $I$ . Here we focus on one of the singular solutions (cf p 5798 in Contopoulos *et al* [6]):

Inserting and identifying coefficients in the Laurent series expansion show that the singular expansions

$$\begin{aligned} X &= \frac{i}{s} + \gamma_1 s + \dots & Y &= x_2 s + \dots & Z &= x_3 s + \dots \\ p_x &= -\frac{2}{s} + b + cs + \dots & p_y &= p_2 + 2ix_2 s + \dots & p_z &= p_3 + 2ix_3 s + \dots \end{aligned} \tag{9}$$

satisfy the set of equations (6) to zeroth order in  $s = t - \tau_0$ , provided  $b = p_2 + p_3$  and  $c = 2i[x_2 + x_3 - 2\gamma_1]$ . Here,  $t$  is the complex time variable and  $\tau_0, \gamma_1, x_2, x_3, p_2$  and  $p_3$  are the six free parameters in the generic Painlevé expansion. Computing  $I$  to zeroth order (all other terms must vanish due to the time invariance of  $I$ ) we get

$$I = p_2 p_3 + 2i(x_2 + x_3 - 3\gamma_1) \tag{10}$$

which is a constant of motion but in general *does not* vanish. Thus we see, as is the case for the other classes of singular solutions in [6], that the set of equations has the ARS Painlevé property independent of the value of the first integral  $I$ .

#### 4. Closed orbits and non-integrability

One way to characterize non-integrability is by means of non-zero Lyapunov exponents. In general, each first integral of a Hamiltonian system implies the vanishing of two Lyapunov exponents due to preservation of symplectic two-forms [21]. In practice this characterization is not very useful. First, it is not easy to establish numerically due to the infinite time limit involved and, second, because the reverse need not be true, i.e. the vanishing of  $2n$  Lyapunov exponents does not imply the existence of  $n$  first integrals. However, in general the vanishing of Lyapunov exponents occurs for almost all trajectories of infinite length. In particular, we shall consider *closed orbits*, i.e. trajectories which return to a given point,  $x$ , in the phase space after some period  $T$ . Periodic orbits have the great advantage (see also [22]) that they are completely determined by their finite-time behaviour and therefore provide a numerically easy and reliable way of stating non-integrability of the system.

We write the differential equations (4)–(6) in the form:

$$\frac{d}{dt}x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla H(x) \equiv J \nabla H(x) \quad x = (p, q) \in R^d \times R^d \tag{11}$$

(here  $d = 3$ ) and we denote by  $\phi_H^t(x)$  the flow obtained by integrating this equation. A real-analytic function  $I$  on  $R^{2d}$  is said to be a first integral if it commutes with  $H$ , i.e. if their Poisson bracket vanishes:

$$\{I, H\} \equiv \langle \nabla I, J \nabla H \rangle \equiv 0 \tag{12}$$

$(\cdot, \cdot)$  being the usual inner product in  $R^{2d}$ . In particular [2], this condition implies that the flow  $\phi_H^t(x)$ , generated by  $J \nabla I$ , commutes with  $\phi_H^s$ . Thus, we have the two relations:

$$I \circ \phi_H^t = I \quad t \in R \quad \phi_H^s \circ \phi_H^t = \phi_H^t \circ \phi_H^s \quad s, t \in R. \tag{13}$$

Assume now that we have found a periodic orbit  $x = \phi_H^T(x)$  of our Hamiltonian flow with return time  $T > 0$ . We may use this periodicity condition in our two relations above, taking derivatives in  $x$  and in  $s$  (at  $s = 0$ ), respectively, to obtain (tr meaning transpose)

$$\nabla I(x)^T D\phi_H^T(x) = \nabla I(x)^T \quad J \nabla I(x) = D\phi_H^T(x) J \nabla I(x). \tag{14}$$

We can replace  $I$  with  $H$  or any other first integral (provided they all commute) and conclude that the (transposed) set of  $\nabla H, \nabla I$ , etc generates a set of left invariant eigenvectors of  $D\phi_H^T(x)$  while by multiplying by  $J$  we obtain a set of right invariant eigenvectors of the same matrix.

The two sets are evidently mutually orthogonal (as the first integrals commute). Thus we may orthonormalize each set and take their union to be a basis of  $R^{2d}$ . A straightforward calculation shows that in this basis  $D\phi_H^T(x)$  takes the form of an upper (or lower) triangular matrix with 1 in the diagonal. Thus, we conclude that if the system has  $d$  independent

first integrals at a periodic orbit then the stability matrix has only one eigenvalue, 1, with degeneracy  $2N$ .

As Poincaré has already noted [23], the presence of non-unit eigenvalues does not exclude integrability but could germinate from a degeneracy of first integrals. The simplest example of this is the stiff pendulum which, when placed at rest in its top position, is in an unstable fixed point. The separatrices [2] through this point separate the phase space into oscillatory modes and two types of rotational modes. The eigenvalues of the stability matrix for the fixed point for any non-zero time are not on the unit circle. The arguments above do not apply to this case as we here have  $\nabla H = 0$  (a special case of degenerating first integrals). More complicated situations arise in integrable systems in higher dimensions when, for closed orbits, the gradients of the first integrals become linearly dependent.

As mentioned above, a rigorous proof would require the presence of a transversal homoclinic intersection [10, 11]. However, we insist that an abundance of unstable periodic orbits strongly indicates non-integrability of the dynamical system, i.e. that it cannot possess  $d$  commuting analytic first integrals which are independent almost everywhere.

### 5. Unstable periodic orbits for $I < 0$

Finding periodic orbits in a low-dimensional dynamical system such as (1) means solving the equation

$$\phi^T(x) = x \tag{15}$$

and is a straightforward, if tedious, task. One starts with random initial conditions and lets the system evolve until a suitable close-return occurs. What is a suitable close-return is as much determined by elapsed time as distance between initial and final values of the variables, due to the high instability of the dynamics. One then uses the initial conditions of the close-return trajectory as a starting guess in an extended Newton search for a solution to (15). This takes the following steps:

1. Choose a Poincaré section, i.e. a  $(2N - 1)$ -dimensional surface that the trajectory will cross. This is rather arbitrary, but a plane through the initial point perpendicular to the initial velocity works well.

2. Integrate the equations of motion including the Jacobian matrix  $M$  (here  $v = J\nabla H$ )

$$\dot{M} = \frac{\partial v(x(t))}{\partial x} M \quad M_{t=0} = 1 \tag{16}$$

until the trajectory crosses the Poincaré section at  $x_f$ , close to the initial point,  $x_i$ .

3. Assume that the trajectory is close to a periodic orbit and solve the linearized version of (15):

$$x = \phi^T(x) \approx \phi^T(x_i) + M(x - x_i) = x_f + M(x - x_i) \tag{17}$$

i.e. solve

$$(1 - M)x = x_f - Mx_i. \tag{18}$$

This presents the problem that since  $M$  has 1 as (at least) a double eigenvalue,  $1 - M$  is not invertible. To remedy this, we augment equation (18) with two more equations and two more variables [24]. The two extra equations serve the purpose of keeping the solution  $x$  on the Poincaré section and preserving the first integral  $I$  during the Newton iteration, while the two variables add corrections to  $x$  along the right/left eigenvectors of the eigenvalue 1, i.e.  $v(x)$  and  $\nabla I$ , in order to ensure the existence of a solution to (18).

4. Repeat until satisfactory convergence is obtained.

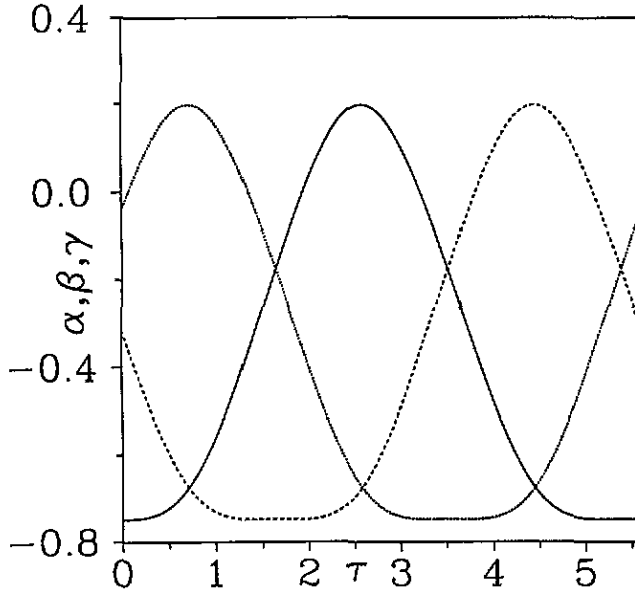


Figure 1. The simplest periodic orbit of the mixmaster system corresponding to entry *i* in table 1. The three scale factors  $\alpha$ ,  $\beta$  and  $\gamma$  each make one identical oscillation. The trajectory takes the form of a slightly distorted circle.

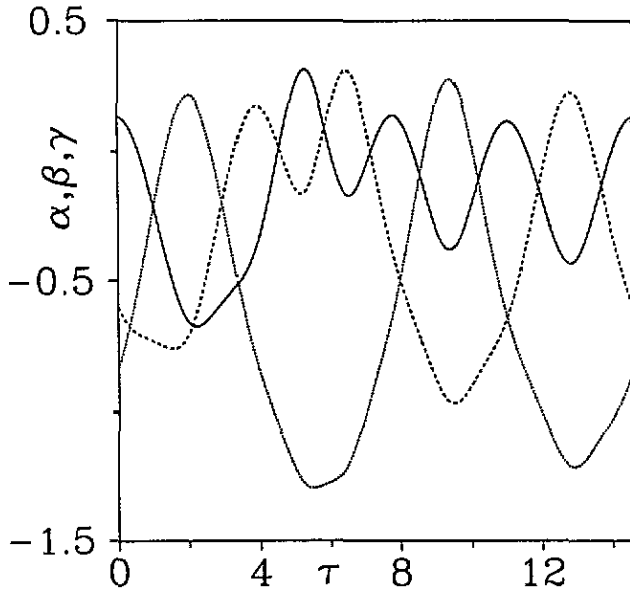


Figure 2. A more complex periodic orbit of the mixmaster system corresponding to entry *v* in table 1. The largest eigenvalue  $\Lambda$  of the Jacobian for this trajectory is  $\sim 5800$  for a period  $T$  of only 14.62.

We plot two examples of periodic orbits in figures 1 and 2. In tables 1 and 2 we give the initial conditions, periods and eigenvalues of a few of the periodic orbits found through

**Table 1.** Initial conditions for some periodic orbits. Entries i, v and vi correspond to figures 1, 2+3 and 4. The number of digits given corresponds to the accuracy of the numerical methods employed as estimated by using two different integration routines and two different tolerance levels in the routines.

No.	Initial values			Period <i>T</i>
	$\alpha, \dot{\alpha}$	$\beta, \dot{\beta}$	$\gamma, \dot{\gamma}$	
i	-0.746 506 413 6 0.0	-0.041 105 930 28 0.573 582 912 96	-0.314 591 361 09 -0.633 033 464 2	5.604 035 044 2
ii	0.280 881 916 55 0.0	-1.404 341 474 8 -0.191 324 482 56	-0.216 446 176 46 0.084 543 111 864	7.947 287 069 9
iii	-0.858 226 542 86 0.0	0.296 817 610 585 0.0	-0.374 109 406 546 0.0	8.544 140 961 78
iv	0.399 801 735 0.0	-0.141 272 123 1.044 336 138	-0.268 969 97 -0.472 639 76	13.413 543 212 8
v	0.136 510 165 17 0.0	-0.833 358 652 26 0.424 450 201 779	-0.598 734 635 8 -0.324 312 858 74	14.626 399 870 8
vi	0.672 156 268 48 0.0	-0.048 513 484 37 0.230 266 776 2	-0.048 513 484 37 -0.230 266 776	16.150 302 71
vii	-1.066 369 624 0 0.0	-2.797 810 588 0 0.066 856 134 9	0.077 800 693 95 0.346 399 983 2	16.435 444 052

**Table 2.** Eigenvalues for the periodic orbits of table 1. The lesser accuracy of the unit eigenvalues is caused by the fact that they appear as double eigenvalues.

	Eigenvalues			
i	-29.654 128 172 -0.033 722 117 682	1.000 00 1.000 00	-0.948 098 014 -0.948 098 014	+i0.317 978 231 7 -i0.317 978 231 7
ii	-103.545 671 685 -9.657 574 128 10 <sup>-3</sup>	1.000 00 1.000 00	-0.697 555 607 23 -0.697 555 607 23	+i0.716 530 651 69 -i0.716 530 651 69
iii	136.451 552 742 7.328 608 432 10 <sup>-3</sup>	1.000 00 1.000 00	-0.487 064 950 12 -0.487 064 950 12	+i0.873 365 750 62 -i0.873 365 750 62
iv	2540.896 082 5 3.935 619 4 10 <sup>-4</sup>	1.000 00 1.000 00	1.044 452 134 0.957 439 758	
v	-5843.077 705 -1.711 426 8 10 <sup>-4</sup>	1.0000 1.0000	0.530 715 831 9 0.530 715 831 9	+i0.847 549 824 9 -i0.847 549 824 9
vi	-643.893 404 6 -1.553 052 09 10 <sup>-3</sup>	1.000 00 1.000 00	0.002 135 740 7 0.002 135 740 7	+i0.999 997 719 3 -i0.999 997 719 3
vii	1926.432 855 64 5.190 941 3 10 <sup>-4</sup>	1.0000 1.0000	2.170 605 581 0.460 700 925 3	

this method. As can be seen, the eigenvalue 1 has multiplicity 2 in all our examples. We have found approximately 200 unstable periodic orbits, of which around 50 were found in a random search through phase space, whereas the remaining number were found in



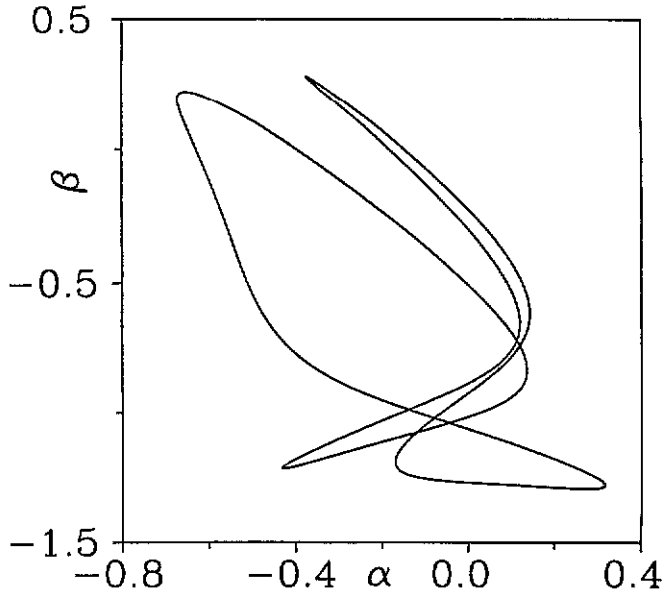


Figure 3. The same trajectory as above but shown projected on the  $(\alpha, \beta)$  plane.

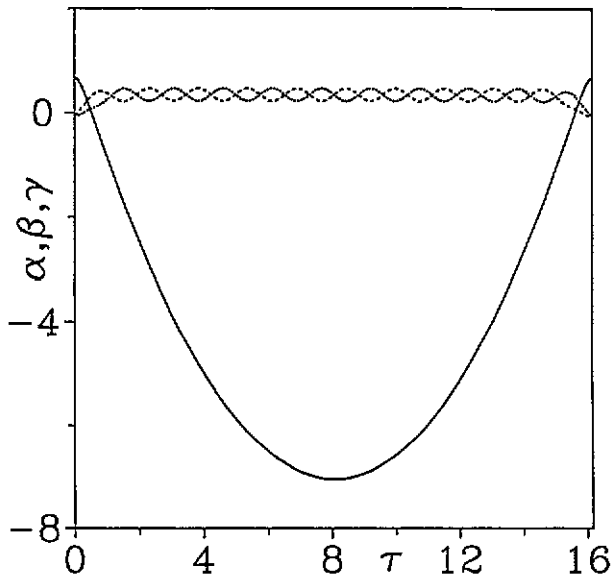


Figure 4. An example of a periodic orbit in the family with respectively 1,  $n$  and  $n$  oscillations of the three scale factors with  $n = 10$ . The initial values correspond to entry vi in table 1. For increasing  $n$  the difference between the two oscillating scale factors becomes smaller and smaller.

a systematic search for members of infinite families of orbits as described below. This numerical result makes it highly unlikely that the system is integrable for  $I < 0$ .

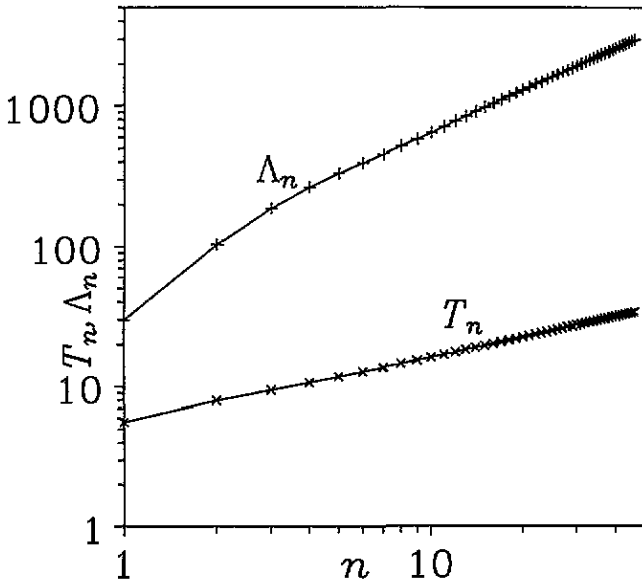
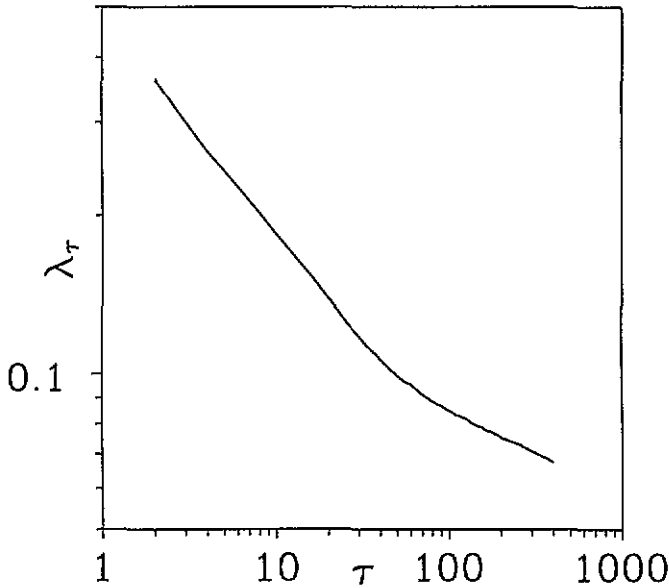


Figure 5. The power-law dependence of the largest instability eigenvalue  $\Lambda$  of the Jacobian,  $M$ , and the period,  $T$ , against  $n$  in the  $(1, n, n)$  family of periodic orbits. We have found orbits up to  $n = 46$ . For  $n \geq 6$  we estimate the slope to be .495 for  $T$  and 1.006 for  $\Lambda$ . Some of the data points are listed in table 3.

Table 3. Largest stability eigenvalue, period and stability exponent for the  $(1, n, n)$  family of periodic orbits.  $T_n$  and  $\Lambda_n$  are plotted against  $n$  in figure 5. Beyond  $n \approx 40$  it gets increasingly difficult to follow the trajectories numerically because the dynamics is determined by the decreasing difference between the two oscillating scale factors  $\beta$  and  $\gamma$ .

$n$	$T$	$\Lambda$	$\lambda \equiv \log( \Lambda )/T$
1	5.604 035 044 1	-29.654 128 172	-0.114 304 535 48
2	7.947 287 069 9	-103.545 671 685 2	-0.044 811 267 454 99
3	9.425 697 274 83	-186.757 808 548	-0.028 003 180 571 8
5	11.701 564 863 2	-331.024 462 37	-0.017 527 986 407 6
10	16.150 302 715	-643.893 404 65	-0.010 044 415 962 2
15	19.649 946 461	-967.114 124 0	-0.007 108 071 670 5
20	22.618 966 753	-1292.236 918 3	-0.005 543 975 674 2
25	25.242 560 127	-1617.915 587	-0.004 566 921 77
30	27.618 644 59	-1943.840 078 0	-0.003 895 598 615 71
40	31.844 037 93	-2596.030 034	-0.003 028 369 681
46	34.129 324 14	-2987.456 574	-0.002 678 592 127

The existence of unstable periodic orbits does not lead directly to chaos in the sense of a positive Lyapunov exponent in the mixmaster system for  $I < 0$ . In fact, we have been able to find series of periodic orbits that indicate that the Lyapunov exponent will converge to zero for infinite time. Consider a family of periodic orbits where one of the scale factors,  $\alpha$ , makes one oscillation while the other two,  $\beta$  and  $\gamma$ , make  $n$  oscillations. In figure 4 such an orbit with  $n = 10$  is shown. We find that for such a family the largest eigenvalue of  $M$ ,  $\Lambda_n$  grows linearly with  $n$ , whereas the period,  $T_n$  grows as  $\sqrt{n}$ . This means that for the instability exponent  $\lambda_n$ , i.e. the local Lyapunov exponent of the individual periodic



**Figure 6.** The finite-time Lyapunov exponent  $\lambda_\tau$  computed as an average of 1000 random trajectories. For numerical reasons we use the largest eigenvalue of  $M^\tau M$ .  $\tau = 70$  is approximately the time limit, where we lose information about the lowest eigenvalue due to numerical underflow. We believe this explains the bend of the curve near this value and therefore that the power-law fall-off holds even for  $\tau > 70$ . The power-law fall-off indicates that the infinite-time Lyapunov exponent is 0.

orbits, we have

$$\lambda_n \equiv \frac{\log \Lambda_n}{T_n} \sim \frac{\log n}{\sqrt{n}} \quad (19)$$

which will tend to zero for  $n$  going to infinity. This is shown in table 3 and figure 5. The limit of such a family of periodic orbits is therefore marginally stable. In the computation of the Lyapunov exponent, marginally stable regions seem to dominate making the exponent converge to zero. This conclusion is supported by a direct computation of the finite-time approximation of the Lyapunov exponent. Starting with 1000 random initial conditions, we have computed the average finite-time Lyapunov exponent, which appears to fall off as a power law with increasing  $\tau$ . This result is shown in figure 6.

We therefore conclude that the mixmaster system (1) for  $I < 0$  is neither integrable nor chaotic in the sense of having a positive maximal Lyapunov exponent.

## 6. Concluding remarks

We have noted that, according to the recent analysis by Contopoulos *et al* [6], the mixmaster cosmological model passes the Painlevé test not only for the case  $I = 0$ , i.e. the vacuum Einstein equations for which the behaviour of the axes is described by the BKL combinatorial model, but for arbitrary values of the first integral  $I$ . At the same time we have shown that when  $I < 0$ , the model is probably non-integrable through the existence of an abundance of unstable periodic orbits.

Thus, the model provides a first example of a Hamiltonian system of nonlinear differential equations that passes the ARS test for the Painlevé property but which is not

integrable. One may still speculate whether the mixmaster model has the full Painlevé property (the answer appears to be negative according to the recent results of Contopoulos *et al* [12] and those of Latifi *et al* [13]), or whether the constraint  $I = 0$  may be incorporated in a dynamical context or in the Painlevé analysis in some way, avoiding the non-integrable periodic solutions described in this article and rendering the system integrable.

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